ON CONWAY-GORDON TYPE THEOREMS FOR GRAPHS IN THE PETERSEN FAMILY

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Abstract. For every spatial embedding of each graph in the Petersen family, it is known that the sum of the linking numbers over all of the constituent 2-component links is congruent to 1 modulo 2. In this paper, we give an integral lift of this formula in terms of the square of the linking number and the second coefficient of the Conway polynomial.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let G be a finite graph. An embedding f of G into the 3-sphere is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph*. We denote the set of all spatial embeddings of G by $\mathrm{SE}(G)$. We call a subgraph γ of G which is homeomorphic to the circle a *cycle* of G and denote the set of all cycles of G by $\Gamma(G)$. In particular, we call a cycle of G which contains exactly k edges a *k-cycle* of G and denote the set of all k-cycles of G by $\Gamma_k(G)$. For a positive integer n, $\Gamma^{(n)}(G)$ denotes the set of all cycles of G $(=\Gamma(G))$ if $n=1$ and the set of all unions of mutually disjoint n cycles of G if $n \geq 2$. We denote the union of $\Gamma^{(n)}(G)$ over all positive integer n by $\bar{\Gamma}(G)$. For an element γ in $\Gamma^{(n)}(G)$ and an element f in $\mathrm{SE}(G)$, $f(\gamma)$ is none other than a knot in $f(G)$ if $n=1$ and an n-component link in $f(G)$ if $n \geq 2$.

A *$\triangle Y$-exchange* is an operation to obtain a new graph G_Y from a graph $G_\triangle$ by removing all edges of a 3-cycle $\triangle = [uvw]$ of $G_\triangle$ with the edges uv, vw and wu, and adding a new vertex x and connecting it to each of the vertices u, v and w as illustrated in Fig. 1.1 (we often denote $ux \cup vx \cup wx$ by Y). A *$Y\triangle$-exchange* is the reverse of this operation. Let K_n be the *complete graph* on n vertices, namely the simple graph consisting of n vertices in which every pair of distinct vertices is connected by exactly one edge. The set of all graphs obtained from K_6 by a finite sequence of $\triangle Y$-exchanges and $Y\triangle$-exchanges is called the *Petersen family*. The Petersen family consists of seven graphs K_6, Q_7, Q_8, P_7, P_8, P_9 and the *Petersen graph* P_{10} as illustrated in Fig. 1.2, where an arrow between two graphs indicates the application of a single $\triangle Y$-exchange. For spatial embeddings of a graph in the Petersen family, the following is known.

Theorem 1.1. *Let G be an element in the Petersen family. For any element f in* $\mathrm{SE}(G)$*, it follows that*

$$\sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma)) \equiv 1 \pmod 2,$$

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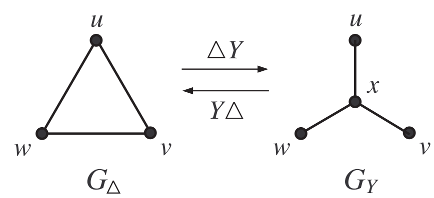

FIGURE 1.1.

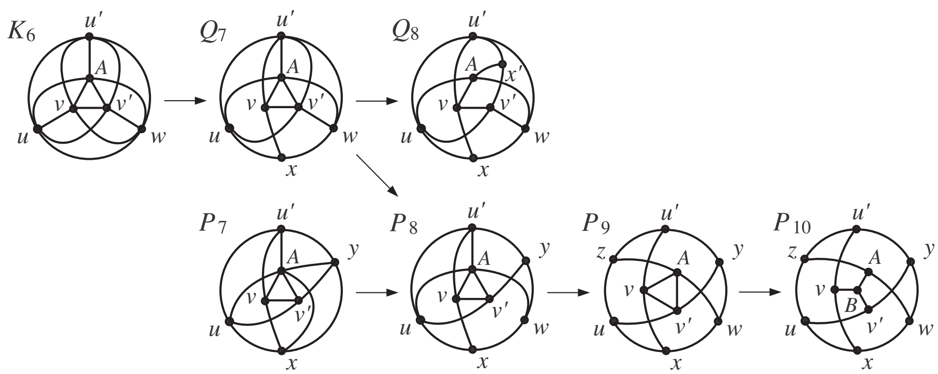

FIGURE 1.2.

where lk *denotes the linking number in the* 3*-sphere.*

We remark here that the case of $G = K_6$ in Theorem 1.1 is what is called the Conway-Gordon K_6 theorem [1], and the other cases were shown by Sachs [6] indirectly, and also pointed out by Taniyama-Yasuhara [7]. Theorem 1.1 implies that each element G in the Petersen family is *intrinsically linked*, that is, for any element f in $\mathrm{SE}(G)$, there exists an element γ in $\Gamma^{(2)}(G)$ such that $f(\gamma)$ is a nonsplittable 2-component link. It is known that a graph is intrinsically linked if and only if the graph contains an element in the Petersen family as a minor [5]. Namely, the Petersen family plays a role of a complete obstruction for graphs not to be intrinsically linked.

Our purpose in this paper it to give an integral lift of Theorem 1.1. In the following, $a_i(L)$ denotes the i-th coefficient of the *Conway polynomial* for an oriented link L.

Theorem 1.2. *Let G be an element in the Petersen family. We give the labels for all vertices of G as indicated in Fig. 1.2. Let ω_G be a map from $\Gamma(G)$ to the set of integers $\mathbb{Z}$ defined as follows:*

(1) *If $G = K_6$, then for an element γ in $\Gamma(K_6)$, we define*

$$\omega_{K_6}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_6(K_6) \\ -1 & \text{if } \gamma \in \Gamma_5(K_6) \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If $G = Q_7$, then for an element γ in $\Gamma(Q_7)$, we define*

$$\omega_{Q_7}(\gamma) = \begin{cases} 1 & \textit{if } \gamma \in \Gamma_7(Q_7) \cup \Gamma_6^1(Q_7) \\ -1 & \textit{if } \gamma \in \Gamma_6^2(Q_7) \cup \Gamma_5(Q_7) \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} \Gamma_6^1(Q_7) &= \{\delta \in \Gamma_6(Q_7) \mid \delta \not\ni x\}, \\ \Gamma_6^2(Q_7) &= \{\delta \in \Gamma_6(Q_7) \mid \delta \supset \{x, u, v, w\}\}. \end{aligned}$$

(3) *If $G = Q_8$, then for an element γ in $\Gamma(Q_8)$, we define*

$$\omega_{Q_8}(\gamma) = \begin{cases} 1 & \textit{if } \gamma \in \Gamma_8(Q_8) \cup \Gamma_6^1(Q_8) \\ -1 & \textit{if } \gamma \in \Gamma_6^2(Q_8) \\ 0 & \text{(otherwise),} \end{cases}$$

where

$$\begin{aligned} \Gamma_6^1(Q_8) &= \{\delta \in \Gamma_6(Q_8) \mid \delta \cap \{x, x'\} = \emptyset\}, \\ \Gamma_6^2(Q_8) &= \{\delta \in \Gamma_6(Q_8) \mid \delta \cap \{x, x'\} \neq \emptyset\}. \end{aligned}$$

(4) *If $G = P_7$, then for an element γ in $\Gamma(P_7)$, we define*

$$\omega_{P_7}(\gamma) = \begin{cases} 1 & \textit{if } \gamma \in \Gamma_7(P_7) \\ -1 & \textit{if } \gamma \in \Gamma_5(P_7) \\ -2 & \textit{if } \gamma \in \Gamma_6^1(P_7) \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\Gamma_6^1(P_7) = \{\delta \in \Gamma_6(P_7) \mid \delta \not\ni A\}.$$

(5) *If $G = P_8$, then for an element γ in $\Gamma(P_8)$, we define*

$$\omega_{P_8}(\gamma) = \begin{cases} 1 & \textit{if } \gamma \in \Gamma_8(P_8) \cup \Gamma_7^1(P_8) \\ -1 & \textit{if } \gamma \in \Gamma_7^2(P_8) \cup \Gamma_6^1(P_8) \cup \Gamma_5(P_8) \\ -2 & \textit{if } \gamma \in \Gamma_6^2(P_8) \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} \Gamma_7^1(P_8) &= \{\delta \in \Gamma_7(P_8) \mid \delta \not\supset \{x, y, w\}\}, \\ \Gamma_7^2(P_8) &= \{\delta \in \Gamma_7(P_8) \mid \delta \not\ni A\}, \\ \Gamma_6^1(P_8) &= \{\delta \in \Gamma_6(P_8) \mid \delta \ni w\}, \\ \Gamma_6^2(P_8) &= \{\delta \in \Gamma_6(P_8) \mid \delta \cap \{A, w\} = \emptyset\}. \end{aligned}$$

(6) *If $G = P_9$, then for an element γ in $\Gamma(P_9)$, we define*

$$\omega_{P_9}(\gamma) = \begin{cases} 1 & \textit{if } \gamma \in \Gamma_9(P_9) \cup \Gamma_8^1(P_9) \\ -1 & \textit{if } \gamma \in \Gamma_7^1(P_9) \cup \Gamma_6^1(P_9) \cup \Gamma_5(P_9) \\ -2 & \textit{if } \gamma \in \Gamma_6^2(P_9) \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned}\Gamma_8^1(P_9) &= \{\delta \in \Gamma_8(P_9) \mid \delta \supset \{A, v, v'\}\},\\ \Gamma_7^1(P_9) &= \{\delta \in \Gamma_7(P_9) \mid \delta \not\supset \{A, v, v'\}\},\\ \Gamma_6^1(P_9) &= \{\delta \in \Gamma_6(P_9) \mid \delta \supset \{A, v, v'\}\},\\ \Gamma_6^2(P_9) &= \{\delta \in \Gamma_6(P_9) \mid \delta \not\supset \{A, v, v'\}\}.\end{aligned}$$

(7) *If $G = P_{10}$, then for an element γ in $\Gamma(P_{10})$, we define*

$$\omega_{P_{10}}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_9(P_{10}) \\ -2 & \text{if } \gamma \in \Gamma_6(P_{10}) \\ -1 & \text{if } \gamma \in \Gamma_5(P_{10}) \\ 0 & \text{otherwise.} \end{cases}$$

Then for any element f in $\mathrm{SE}(G)$, it follows that

$$2 \sum_{\gamma \in \Gamma(G)} \omega_G(\gamma) a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma))^2 - 1.$$

Note that Theorem 1.1 can be obtained from Theorem 1.2 by taking the modulo two reduction. We also should remark here that Theorem 1.2 has already known in the case of $G = K_6$ (Nikkuni [2]), P_7 (O'Donnol [4]) and Q_7 (Nikkuni-Taniyama [3]). The other cases are new.

We say that an element f in $\mathrm{SE}(G)$ is *knotted* if $f(G)$ contains a nontrivial knot, and *complexly algebraically linked* if $f(G)$ contains a 2-component link whose linking number is not equal to $0, \pm 1$ or a pair of 2-component links with nonzero linking number [4]. Then O'Donnol showed the following.

Theorem 1.3. (O'Donnol [4]) *Let G be an element in the Petersen family and f an element in $\mathrm{SE}(G)$. If f is complexly algebraically linked, then f is knotted.*

In [4], the place of the cycle whose image is a nontrivial knot was not examined. As an application of Theorem 1.2, we give an alternative proof of Theorem 1.3 in more refined form as follows.

Corollary 1.4. *Let G be an element in the Petersen family and ω_G a map from $\Gamma(G)$ to $\mathbb{Z}$ in Theorem 1.2. Let f be an element in $\mathrm{SE}(G)$. If f is complexly algebraically linked, then it follows that*

$$\sum_{\gamma \in \Gamma(G)} \omega_G(\gamma) a_2(f(\gamma)) \geq 1.$$

Proof. By Theorem 1.1, it follows that f is complexly algebraically linked if and only if

$$\sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma))^2 \geq 3. \tag{1.1}$$

Then by Theorem 1.2 and (1.1), it follows that

$$\sum_{\gamma \in \Gamma(G)} \omega_G(\gamma) a_2(f(\gamma)) = \frac{1}{2} \left\{ \sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma))^2 - 1 \right\} \geq 1.$$

Thus we have the desired conclusion. $\square$

By Corollary 1.4, there exists an element γ_0 in $\Gamma(G)$ with $\omega_G(\gamma_0) \neq 0$ such that $a_2(f(\gamma_0)) \neq 0$. Namely, Corollary 1.4 refines Theorem 1.3 at a point of identifying the place of nontrivial knots in $f(G)$.

In the next section, we prepare general results for $\triangle Y$-exchanges and the Conway-Gordon type theorems which are based on [3]. We give a proof of Theorem 1.2 in section 3.

2. General results

Let $G_\triangle$ and G_Y be two graphs such that G_Y is obtained from $G_\triangle$ by a single $\triangle Y$-exchange. We denote the set of all elements in $\bar{\Gamma}(G_\triangle)$ containing $\triangle$ by $\bar{\Gamma}_\triangle(G_\triangle)$. Let γ' be an element in $\bar{\Gamma}(G_\triangle)$ which does not contain $\triangle$. Then there exists an element $\bar{\Phi}(\gamma')$ in $\bar{\Gamma}(G_Y)$ such that $\gamma' \setminus \triangle = \bar{\Phi}(\gamma') \setminus Y$. It is easy to see that the correspondence from γ' to $\bar{\Phi}(\gamma')$ defines a surjective map

$$\bar{\Phi} = \bar{\Phi}_{G_\triangle, G_Y} : \bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle) \longrightarrow \bar{\Gamma}(G_Y). \tag{2.1}$$

In particular, if γ' is an element in $\Gamma^{(n)}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)$ then $\bar{\Phi}(\gamma')$ is an element in $\Gamma^{(n)}(G_Y)$. This implies that the restriction map of $\bar{\Phi}$ on $\Gamma^{(n)}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)$ induces a surjective map from $\Gamma^{(n)}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)$ to $\Gamma^{(n)}(G_Y)$, and thus it is clear that if $\Gamma^{(n)}(G_\triangle)$ is an empty set, then $\Gamma^{(n)}(G_Y)$ is also an empty set. The inverse image of an element γ in $\bar{\Gamma}(G_Y)$ by $\bar{\Phi}$ contains at most two elements in $\bar{\Gamma}(G_\triangle) \setminus \bar{\Gamma}_\triangle(G_\triangle)$. Fig. 2.1 illustrates the case that the inverse image of γ by $\bar{\Phi}$ consists of exactly two elements. In general, the inverse image of γ by $\bar{\Phi}$ consists of exactly one element if and only if γ contains u, v, w and x, or γ does not contain x.

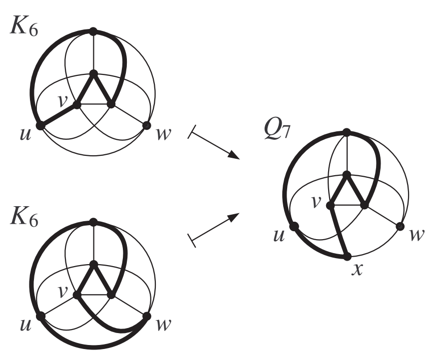

Figure 2.1.

Let A be an additive group. We say that an A-valued unoriented link invariant α is *compressible* if $\alpha(L) = 0$ for any unoriented link L which have a component K bounding a disk D in the 3-sphere with $D \cap L = \partial D = K$. Suppose that for each element γ' in $\bar{\Gamma}(G_\triangle)$, an A-valued unoriented link invariant $\alpha_{\gamma'}$ is assigned. Then for each element γ in $\bar{\Gamma}(G_Y)$, we define an A-valued unoriented link invariant $\tilde{\alpha}_\gamma$ by

$$\tilde{\alpha}_\gamma(L) = \sum_{\gamma' \in \bar{\Phi}^{-1}(\gamma)} \alpha_{\gamma'}(L)$$

for an unoriented link L. Then the following theorem holds.

Theorem 2.1. (Nikkuni-Taniyama [3]) *Suppose that $\alpha_{\gamma'}$ is compressible for each element γ' in $\bar{\Gamma}_{\triangle}(G_{\triangle})$. Suppose that there exists a fixed element c in A such that*

$$\sum_{\gamma' \in \bar{\Gamma}(G_{\triangle})} \alpha_{\gamma'}(g(\gamma')) = c$$

for any element g in $\mathrm{SE}(G_{\triangle})$. Then we have

$$\sum_{\gamma \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_{\gamma}(f(\gamma)) = c$$

for any element f in $\mathrm{SE}(G_Y)$.

By an application of Theorem 2.1, the following is shown.

Theorem 2.2. (Nikkuni-Taniyama [3]) *Let G be an element in the Petersen family. Then, there exists a map ω_G from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element f in $\mathrm{SE}(G)$, it follows that*

$$2 \sum_{\gamma \in \Gamma(G)} \omega_G(\gamma) a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\gamma))^2 - 1.$$

We give a proof of Theorem 2.2 for reader's convinience.

Proof of Theorem 2.2. Let $G_{\triangle}$ and G_Y be two elements in the Petersen family such that G_Y is obtained from $G_{\triangle}$ by a single $\triangle Y$-exchange. Assume that there exists a map ω from $\bar{\Gamma}(G_{\triangle})$ to $\mathbb{Z}$ such that for any element g in $\mathrm{SE}(G_{\triangle})$, it follows that

$$2 \sum_{\gamma' \in \Gamma(G_{\triangle})} \omega(\gamma') a_2(g(\gamma')) = \sum_{\gamma' \in \Gamma^{(2)}(G_{\triangle})} \mathrm{lk}(g(\gamma'))^2 - 1. \tag{2.2}$$

For each element γ' in $\bar{\Gamma}(G_{\triangle})$, we define an integer-valued unoriented link invariant $\alpha_{\gamma'}$ of an unoriented link L as follows. Note that $G_{\triangle}$ is obtained from K_6 or P_7 by a finite sequence of $\triangle Y$-exchanges. Since both $\Gamma^{(n)}(K_6)$ and $\Gamma^{(n)}(P_7)$ are empty sets for $n \geq 3$, we have $\Gamma^{(n)}(G_{\triangle})$ is an empty set for $n \geq 3$. If γ' is an element in $\Gamma(G_{\triangle})$, then $\alpha_{\gamma'}(L) = 2\omega(\gamma')a_2(L)$ if L is a knot and 0 if L is not a knot. If γ' is an element in $\Gamma^{(2)}(G_{\triangle})$, then $\alpha_{\gamma'}(L) = -\mathrm{lk}(L)^2$ if L is a 2-component link and 0 if L is not a 2-component link. Then by (2.2), we have

$$\sum_{\gamma' \in \bar{\Gamma}(G_{\triangle})} \alpha_{\gamma'}(g(\gamma')) = -1. \tag{2.3}$$

Note that $\alpha_{\gamma'}$ is compressible for any element γ' in $\bar{\Gamma}(G_{\triangle})$. Thus by Theorem 2.1 and (2.3), for any element f in $\mathrm{SE}(G_Y)$, it follows that

$$\sum_{\gamma \in \bar{\Gamma}(G_Y)} \tilde{\alpha}_{\gamma}(f(\gamma)) = -1. \tag{2.4}$$

Now we define a map $\tilde{\omega}$ from $\Gamma(G_Y)$ to $\mathbb{Z}$ by

$$\tilde{\omega}(\gamma) = \sum_{\gamma' \in \bar{\Phi}^{-1}(\gamma)} \omega(\gamma') \tag{2.5}$$

for an element γ in $\Gamma(G_Y)$. Let γ be an element in $\bar{\Gamma}(G_Y)$. Note that $\Gamma^{(n)}(G_Y)$ is also an empty set for $n \geq 3$. If γ is belong to $\Gamma(G_Y)$, then by (2.5), we have

$$\tilde{\alpha}_{\gamma}(f(\gamma)) = 2 \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma') a_2(f(\gamma)) = 2\tilde{\omega}(\gamma) a_2(f(\gamma)). \tag{2.6}$$

If γ is belong to $\Gamma^{(2)}(G_Y)$, then $\bar{\Phi}^{-1}(\gamma)$ consists of exactly one element because each union of mutually disjoint two cycles of a graph in the Petersen family contains all of the vertices of the graph. Then we have

$$\tilde{\alpha}_\gamma(f(\gamma)) = \alpha_{\bar{\Phi}^{-1}(\gamma)}(f(\gamma)) = -a_1(f(\gamma))^2. \tag{2.7}$$

Thus by combining (2.4), (2.6) and (2.7), we have

$$2\sum_{\gamma\in\Gamma(G_Y)} \tilde{\omega}(\gamma)a_2(f(\gamma)) - \sum_{\gamma\in\Gamma^{(2)}(G_Y)} \mathrm{lk}(f(\gamma))^2 = -1. \tag{2.8}$$

Namely, for any element f in $\mathrm{SE}(G_Y)$, it follows that

$$2\sum_{\gamma\in\Gamma(G_Y)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = \sum_{\gamma\in\Gamma^{(2)}(G_Y)} \mathrm{lk}(f(\gamma))^2 - 1.$$

As we remarked before, the cases of K_6 and P_7 have already shown by [2] and [4], respectively. Thus by repeating the argument as above, we have the desired conclusion. □

In the following, we show a lemma which are useful to prove Theorem 1.2. We say that two cycles of a graph are *edge-disjoint* if the intersection of them does not contain an edge. Let G be a graph and $\triangle_1, \triangle_2, \ldots, \triangle_k$ 3-cycles of G such that $\triangle_i \cap \triangle_j$ is edge-disjoint for $i \neq j$. Then we also can regard $\triangle_i$ as a 3-cycle of the graph obtained from G by a finite sequence of $\triangle Y$-exchanges at $\triangle_j$'s for $i \neq j$. Let G_l be a graph obtained from G_{l-1} by a single $\triangle Y$-exchange at $\triangle_l$, where $G = G_0$ ($l = 1, 2, \ldots, k$). On the other hand, let σ be a permutation of order k. Let G'_l be a graph obtained from G'_{l-1} by a single $\triangle Y$-exchange at $\triangle_{\sigma(l)}$, where $G = G'_0$ ($l = 1, 2, \ldots, k$). Note that $G_k = G'_k$.

Lemma 2.3. *For any element γ in $\bar{\Gamma}(G_k)$, it follows that*

$$\begin{aligned}
&(\bar{\Phi}_{G_{k-1},G_k} \circ \bar{\Phi}_{G_{k-2},G_{k-1}} \circ \cdots \circ \bar{\Phi}_{G_0,G_1})^{-1}(\gamma) \\
= \ &(\bar{\Phi}_{G'_{k-1},G'_k} \circ \bar{\Phi}_{G'_{k-2},G'_{k-1}} \circ \cdots \circ \bar{\Phi}_{G'_0,G'_1})^{-1}(\gamma).
\end{aligned}$$

Proof. Let γ' be an element in the inverse image of γ by $\bar{\Phi}_{G_{k-1},G_k} \circ \bar{\Phi}_{G_{k-2},G_{k-1}} \circ \cdots \circ \bar{\Phi}_{G_0,G_1}$. Since $\triangle_i \cap \triangle_j$ is edge-disjoint for $i \neq j$, we have

$$\begin{aligned}
\gamma &= \bar{\Phi}_{G_{k-1},G_k} \circ \bar{\Phi}_{G_{k-2},G_{k-1}} \circ \cdots \circ \bar{\Phi}_{G_0,G_1}(\gamma') \\
&= \bar{\Phi}_{G'_{k-1},G'_k} \circ \bar{\Phi}_{G'_{k-2},G'_{k-1}} \circ \cdots \circ \bar{\Phi}_{G'_0,G'_1}(\gamma').
\end{aligned}$$

Thus γ' also be an element in the inverse image of γ by $\bar{\Phi}_{G'_{k-1},G'_k} \circ \bar{\Phi}_{G'_{k-2},G'_{k-1}} \circ \cdots \circ \bar{\Phi}_{G'_0,G'_1}$. This implies the result. □

We assign an A-valued unoriented link invariant $\alpha_{\gamma'}$ for each element γ' in $\bar{\Gamma}(G)$. Then for an element γ in $\bar{\Gamma}(G_l)$, we define an A-valued unoriented link invariant $\alpha^{(l)}_\gamma$ by $\alpha^{(l)}_\gamma = \alpha_\gamma$ if $l = 0$ and $\alpha^{(l)}_\gamma = \widetilde{\alpha^{(l-1)}}_\gamma$ if $l = 1, 2, \ldots, k$ with respect to the sequence of $\triangle Y$-exchanges at $\triangle_1, \triangle_2, \ldots, \triangle_k$. On the other hand, we define an A-valued unoriented link invariant $\beta^{(l)}_\gamma$ by $\beta^{(l)}_\gamma = \alpha_\gamma$ if $l = 0$ and $\beta^{(l)}_\gamma = \widetilde{\beta^{(l-1)}}_\gamma$ if $l = 1, 2, \ldots, k$ with respect to the sequence of $\triangle Y$-exchanges at $\triangle_{\sigma(1)}, \triangle_{\sigma(2)}, \ldots, \triangle_{\sigma(k)}$. Then we have the following.

Lemma 2.4. *For any element γ in $\bar{\Gamma}(G_k)$, it follows that $\alpha^{(k)}_\gamma = \beta^{(k)}_\gamma$.*

Proof. Let γ be an element in $\bar{\Gamma}(G_k)$. Then for an unoriented link L, by Lemma 2.3, we have

$$\begin{aligned}\alpha_\gamma^{(k)}(L) &= \sum_{\gamma'\in(\bar{\Phi}_{G_{k-1},G_k}\circ\bar{\Phi}_{G_{k-2},G_{k-1}}\circ\cdots\circ\bar{\Phi}_{G_0,G_1})^{-1}(\gamma)} \alpha_{\gamma'}(L)\\ &= \sum_{\gamma'\in(\bar{\Phi}_{G'_{k-1},G'_k}\circ\bar{\Phi}_{G'_{k-2},G'_{k-1}}\circ\cdots\circ\bar{\Phi}_{G'_0,G'_1})^{-1}(\gamma)} \alpha_{\gamma'}(L)\\ &= \beta_\gamma^{(k)}(L).\end{aligned}$$

Thus we have the result. □

3. Proof of Theorem 1.2

Let G be a graph and $T = \{\triangle_1, \triangle_2, \ldots, \triangle_k\}$ the set of mutually edge-disjoint 3-cycles of G. We say that T is *stable* if for any l-element subset $\{\triangle_{i_1}, \triangle_{i_2}, \ldots, \triangle_{i_l}\}$ of T $(1 \le l < k)$, $\triangle Y$-exchanges at $\triangle_{i_1}, \triangle_{i_2}, \ldots, \triangle_{i_l}$ produce the same graph up to isomorphism.

Proof of Theorem 1.2. We denote the 3-cycles $[uvw]$, $[u'v'w]$, $[uu'A]$, $[vv'A]$ and $[u'v'A]$ of K_6 by $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ and $\triangle_5$, respectively. Note that $\triangle_i \cap \triangle_j$ is edge-disjoint for $1 \le i < j \le 4$, and $\triangle_1 \cap \triangle_5$ is also edge-disjoint.

(1) Let G be K_6. Then this case has been already shown in [2].

(2) Let G be Q_7 which is obtained from K_6 by a single $\triangle Y$-exchange at $\triangle_1$. Though this case has been already shown in [3], we give it again for reader's convenience. Let ω_{K_6} be the map from $\Gamma(K_6)$ to $\mathbb{Z}$ as in (1) and $\tilde{\omega}_{K_6}$ the map from $\Gamma(Q_7)$ to $\mathbb{Z}$ defined by (2.5) with respect to the $\triangle Y$-exchange at $\triangle_1$. In the following we show $\tilde{\omega}_{K_6} = \omega_{Q_7}$. Let γ be an element in $\Gamma(Q_7)$. If γ belongs to $\Gamma_7(Q_7)$, then there uniquely exists an element γ' in $\Gamma_6(K_6)$ such that $\bar{\Phi}^{-1}_{K_6,Q_7}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{K_6}(\gamma) = \omega_{K_6}(\gamma') = 1$. If γ belongs to $\Gamma_6(Q_7)$, then γ does not contain exactly one of the vertices of Q_7. Then it is sufficient to consider the following three cases up to symmetry of Q_7: (i) If γ does not contain x, then there uniquely exists an element γ' in $\Gamma_6(K_6)$ such that $\bar{\Phi}^{-1}_{K_6,Q_7}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{K_6}(\gamma) = \omega_{K_6}(\gamma') = 1$. (ii) If γ does not contain u, then there exists an element γ'_1 in $\Gamma_5(K_6)$ and an element γ'_2 in $\Gamma_6(K_6)$ such that $\bar{\Phi}^{-1}_{K_6,Q_7}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Thus we have $\tilde{\omega}_{K_6}(\gamma) = \omega_{K_6}(\gamma'_1) + \omega_{K_6}(\gamma'_2) = -1 + 1 = 0$. (iii) If γ does not contain u', then there uniquely exists an element γ' in $\Gamma_5(K_6)$ such that $\bar{\Phi}^{-1}_{K_6,Q_7}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{K_6}(\gamma) = \omega_{K_6}(\gamma') = -1$. If γ belongs to $\Gamma_5(Q_7)$, then the inverse image of γ by $\bar{\Phi}$ consists of exactly one element in $\Gamma_5(K_6)$ or a pair of an element in $\Gamma_5(K_6)$ and an element in $\Gamma_4(K_6)$. Thus in any case we have $\tilde{\omega}_{K_6}(\gamma) = -1$. If γ belongs to $\Gamma(Q_7) \setminus \cup_{k=5}^{7}\Gamma_k(Q_7)$, we have $\tilde{\omega}_{K_6}(\gamma) = 0$. In conclusion, we see that

$$\tilde{\omega}_{K_6}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_7(Q_7) \cup \{\delta \in \Gamma_6(Q_7) \mid \delta \not\ni x\} \\ -1 & \text{if } \gamma \in \{\delta \in \Gamma_6(Q_7) \mid \delta \ni x, u, v, w\} \cup \Gamma_5(Q_7) \\ 0 & \text{otherwise} \end{cases}$$

for an element γ in $\Gamma(Q_7)$. Thus it follows that $\tilde{\omega}_{K_6} = \omega_{Q_7}$.

(3) Let G be Q_8 which is obtained from Q_7 by a single $\triangle Y$-exchange at $\triangle_5$. Let ω_{Q_7} be the map from $\Gamma(Q_7)$ to $\mathbb{Z}$ as in (2) and $\tilde{\omega}_{Q_7}$ the map from $\Gamma(Q_8)$ to $\mathbb{Z}$ defined by (2.5) with respect to the $\triangle Y$-exchange at $\triangle_5$. In the following we show $\tilde{\omega}_{Q_7} = \omega_{Q_8}$. Let γ be an element in $\Gamma(Q_8)$. Note that $\Gamma_k(Q_8) = \emptyset$ if $k \neq 4, 6, 8$. If γ belongs to $\Gamma_8(Q_8)$, then there uniquely exists an element γ' in $\Gamma_7(Q_7)$ such

that $\bar{\Phi}^{-1}_{Q_7,Q_8}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{Q_7}(\gamma) = \omega_{Q_7}(\gamma') = 1$. If γ belongs to $\Gamma_6(Q_8)$, then γ does not contain exactly two of the vertices of Q_8. Then it is sufficient to consider the following three cases up to symmetry of Q_8: (i) If γ does not contain either x or x', then there uniquely exists an element γ' in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,Q_8}(\gamma) = \{\gamma'\}$. Since γ' does not contain x, we have $\tilde{\omega}_{Q_7}(\gamma) = \omega_{Q_7}(\gamma') = 1$. (ii) If γ contains exactly one of x and x', then we may assume that γ contains x' by the following reason. Note that $\{\triangle_1, \triangle_5\}$ is stable. Actually, the graph Q_7' which is obtained from K_6 by a single $\triangle Y$-exchange at $\triangle_5$ is isomorphic to Q_7, and Q_8 is also obtained from Q_7' by a single $\triangle Y$-exchange at $\triangle_1$, see Fig. 3.1. Then the map from $\Gamma(Q_8)$ to $\mathbb{Z}$ obtained from ω_{K_6} by $\bar{\Phi}_{Q_7',Q_8} \circ \bar{\Phi}_{K_6,Q_7'}$ coincides with $\tilde{\omega}_{Q_7}$ by (2.6) and Lemma 2.4. Thus the case that γ contains x and the case that γ contains x' are compatible through the isomorphism between Q_7' and Q_7. From now on, to simplify the description of the proof we often use the argument of such a compatibility without any notice. Now assume that γ contains x'. Then there uniquely exists an element γ' in $\Gamma_5(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,Q_8}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{Q_7}(\gamma) = \omega_{Q_7}(\gamma') = -1$. (iii) If γ contains both x and x', then we may assume that γ does not contain either u or u'. Then there exists an element γ_1' in $\Gamma_5(Q_7)$ and an element γ_2' in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,Q_8}(\gamma) = \{\gamma_1', \gamma_2'\}$. Since γ_2' does not contain u, we have $\tilde{\omega}_{Q_7}(\gamma) = \omega_{Q_7}(\gamma_1') + \omega_{Q_7}(\gamma_2') = -1 + 0 = -1$. If γ belongs to $\Gamma_4(Q_8)$, we have $\tilde{\omega}_{Q_7}(\gamma) = 0$. In conclusion, we see that

$$\tilde{\omega}_{Q_7}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_8(Q_8) \cup \{\delta \in \Gamma_6(Q_8) \mid \delta \cap \{x, x'\} = \emptyset\} \\ -1 & \text{if } \gamma \in \{\delta \in \Gamma_6(Q_8) \mid \delta \cap \{x, x'\} \neq \emptyset\} \\ 0 & \text{otherwise} \end{cases}$$

for an element γ in $\Gamma(Q_8)$. Thus it follows that $\tilde{\omega}_{Q_7} = \omega_{Q_8}$.

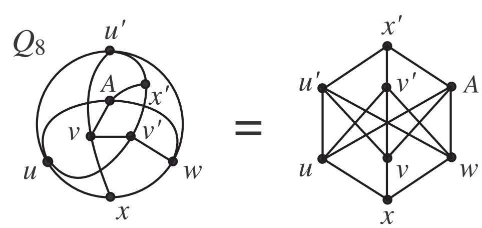

FIGURE 3.1.

(4) Let G be P_7. Then this case has been already shown in [4].

(5) Let G be P_8 which is obtained from Q_7 by a single $\triangle Y$-exchange at $\triangle_2$. Note that $\{\triangle_1, \triangle_2\}$ is stable, see Fig. 3.2. Let ω_{Q_7} be the map from $\Gamma(Q_7)$ to $\mathbb{Z}$ as in (2) and $\tilde{\omega}_{Q_7}$ the map from $\Gamma(P_8)$ to $\mathbb{Z}$ defined by (2.5) with respect to the $\triangle Y$-exchange at $\triangle_2$. In the following we show $\tilde{\omega}_{Q_7} = \omega_{P_8}$. Let γ be an element in $\Gamma(P_8)$. If γ belongs to $\Gamma_8(P_8)$, then there uniquely exists an element γ' in $\Gamma_7(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{Q_7}(\gamma) = \omega_{Q_7}(\gamma') = 1$. If γ belongs to $\Gamma_7(P_8)$, then γ does not contain exactly one of the vertices of P_8. Then it is sufficient to consider the following four cases up to symmetry of P_8: (i) If γ does not contain one of x and y, then we may assume that γ contains x. Then there uniquely exists an element γ' in $\Gamma_7(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma) = \{\gamma'\}$. Thus we have

$\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma')=1$. (ii) If γ does not contain w, then there exists an element γ'_1 in $\Gamma_6(Q_7)$ and an element γ'_2 in $\Gamma_7(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'_1,\gamma'_2\}$. Since γ'_1 also does not contain w, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma'_1)+\omega_{Q_7}(\gamma'_2)=0+1=1$. (iii) If γ does not contain one of u,v,u' and v' (in other words, γ contains all of x,y,w and A), then we may assume that γ does not contain u. Then there uniquely exists an element γ' in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'\}$. Since γ' also does not contain u, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma')=0$. (iv) If γ does not contain A, then there uniquely exists an element γ' in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'\}$. Since γ' contains all of u,v,w and x, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma')=-1$. If γ belongs to $\Gamma_6(P_8)$, then γ does not contain exactly two of the vertices of P_8. Then it is sufficient to consider the following four cases up to symmetry of P_8: (i) If γ contains w and does not contain one of x and y, then we may assume that γ does not contain y. Then there uniquely exists an element γ' in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'\}$. Since γ' contains all of u,v,w and x, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma')=-1$. (ii) If γ does not contain w and one of x and y, then we may assume that γ does not contain y. Then there uniquely exists an element γ' in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'\}$. Since γ' does not contain w, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma')=0$. (iii) If γ does not contain either w or A, then there exists an element γ'_1 in $\Gamma_5(Q_7)$ and an element γ'_2 in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'_1,\gamma'_2\}$. Since γ'_2 contains all of u,v,w and x, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma'_1)+\omega_{Q_7}(\gamma'_2)=-1-1=-2$. (iv) If γ contains all of x,y and w, then we may assume that γ does not contain either u or u'. Then there exists an element γ'_1 in $\Gamma_5(Q_7)$ and an element γ'_2 in $\Gamma_6(Q_7)$ such that $\bar{\Phi}^{-1}_{Q_7,P_8}(\gamma)=\{\gamma'_1,\gamma'_2\}$. Since γ'_2 does not contain u, we have $\tilde{\omega}_{Q_7}(\gamma)=\omega_{Q_7}(\gamma'_1)+\omega_{Q_7}(\gamma'_2)=-1+0=-1$. If γ belongs to $\Gamma_5(P_8)$, then we have $\tilde{\omega}_{Q_7}(\gamma)=-1$ in the same way as the case of Q_7. If γ belongs to $\Gamma(P_8)\setminus\cup_{k=5}^{8}\Gamma_k(P_8)$, we have $\tilde{\omega}_{Q_7}(\gamma)=0$. In conclusion, we see that

$$\tilde{\omega}_{Q_7}(\gamma)=\begin{cases}1 & \text{if } \gamma\in\Gamma_8(P_8)\cup\{\delta\in\Gamma_7(P_8)\mid\delta\not\supset\{x,y,w\}\}\\ -1 & \text{if } \gamma\in\{\delta\in\Gamma_7(P_8)\mid\delta\not\ni A\}\cup\{\delta\in\Gamma_6(P_8)\mid\delta\ni w\}\cup\Gamma_5(P_8)\\ -2 & \text{if } \gamma\in\{\delta\in\Gamma_6(P_8)\mid\delta\cap\{A,w\}=\emptyset\}\\ 0 & \text{otherwise}\end{cases}$$

for an element γ in $\Gamma(P_8)$. Thus it follows that $\tilde{\omega}_{Q_7}=\omega_{P_8}$.

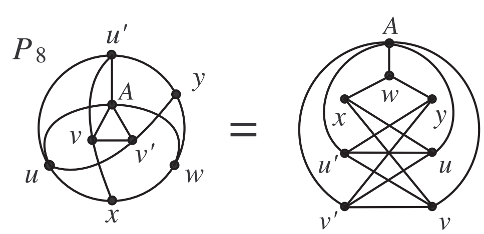

FIGURE 3.2.

(6) Let G be P_9 which is obtained from P_8 by a single $\triangle Y$-exchange at $\triangle_3$. Note that $\{\triangle_1,\triangle_2,\triangle_3\}$ is stable. Let ω_{P_8} be the map from $\Gamma(P_8)$ to $\mathbb{Z}$ as in (5) and $\tilde{\omega}_{P_8}$ the map from $\Gamma(P_9)$ to $\mathbb{Z}$ defined by (2.5) with respect to the $\triangle Y$-exchange at $\triangle_3$. In the following we show $\tilde{\omega}_{P_8}=\omega_{P_9}$. Let γ be an element in $\Gamma(P_9)$. If

γ belongs to $\Gamma_9(P_9)$, then there uniquely exists an element γ' in $\Gamma_8(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = 1$. If γ belongs to $\Gamma_8(P_9)$, then γ does not contain exactly one of the vertices of P_9. Then it is sufficient to consider the following two cases up to symmetry of P_9: (i) If γ contains all of v, v' and A, then we may assume that γ does not contain one of x and w. In any case, there uniquely exists an element γ' in $\Gamma_7(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Since γ' also does not contain one of x and w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = 1$. (ii) If γ does not contain one of v, v' and A, then we may assume that γ does not contain v'. Then there uniquely exists an element γ' in $\Gamma_7(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Since γ' contains all of x, y and w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = 0$. If γ belongs to $\Gamma_7(P_9)$, then γ does not contain exactly two of the vertices of P_9. Then it is sufficient to consider the following two cases up to symmetry of P_9: (i) If γ contains all of v, v' and A, then we may assume that γ does not contain either x or w. Then there uniquely exists an element γ' in $\Gamma_6(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Since γ' contains A and does not contain w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = 0$. (ii) If γ does not contain one of v, v' and A, then we may assume that γ does not contain either x or v, or γ does not contain either u' or v. In the former case, there uniquely exists an element γ' in $\Gamma_6(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Since γ' contains w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = -1$. In the latter case, there exists an element γ'_1 in $\Gamma_6(P_8)$ and an element γ'_2 in $\Gamma_7(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Since γ'_1 contains w and γ'_2 contains all of x, y and w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma'_1) + \omega_{P_8}(\gamma'_2) = -1 + 0 = -1$. If γ belongs to $\Gamma_6(P_9)$, then γ does not contain exactly three of the vertices of P_9. Then it is sufficient to consider the following two cases up to symmetry of P_9: (i) If γ contains all of v, v' and A, then we may assume that γ does not contain any of x, y or w, or γ does not contain any of u, u' or z. In the former case, there uniquely exists an element γ' in $\Gamma_5(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = -1$. In the latter case, there uniquely exists an element γ' in $\Gamma_6(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'\}$. Since γ' contains w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma') = -1$. (ii) If γ does not contain one of v, v' and A, then we may assume that γ does not contain any of x, u' or v, or γ does not contain any of v, v' or A. In any case, there exists an element γ'_1 in $\Gamma_5(P_8)$ and an element γ'_2 in $\Gamma_6(P_8)$ such that $\bar{\Phi}^{-1}_{P_8,P_9}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Since γ'_2 contains w, we have $\tilde{\omega}_{P_8}(\gamma) = \omega_{P_8}(\gamma'_1) + \omega_{P_8}(\gamma'_2) = -1 - 1 = -2$. If γ belongs to $\Gamma_5(P_9)$, then we have $\tilde{\omega}_{P_8}(\gamma) = -1$ in the same way as the case of Q_7. If γ belongs to $\Gamma(P_9) \setminus \cup_{k=5}^{9}\Gamma_k(P_9)$, we have $\tilde{\omega}_{P_8}(\gamma) = 0$. In conclusion, we see that

$$\begin{aligned} &\omega_{P_9}(\gamma) \\ = &\begin{cases} 1 & \text{if } \gamma \in \Gamma_9(P_9) \cup \{\delta \in \Gamma_8(P_9) \mid \delta \supset \{A, v, v'\}\} \\ -1 & \text{if } \gamma \in \{\delta \in \Gamma_7(P_9) \mid \delta \not\supset \{A, v, v'\}\} \cup \{\delta \in \Gamma_6(P_9) \mid \delta \supset \{A, v, v'\}\} \cup \Gamma_5(P_9) \\ -2 & \text{if } \gamma \in \{\delta \in \Gamma_6(P_9) \mid \delta \not\supset \{A, v, v'\}\} \\ 0 & \textit{otherwise} \end{cases} \end{aligned}$$

for an element γ in $\Gamma(P_9)$. Thus it follows that $\tilde{\omega}_{P_8} = \omega_{P_9}$.

(7) Let G be P_{10} which is obtained from P_9 by a single $\triangle Y$-exchange at $\triangle_4$. Note that $\{\triangle_1, \triangle_2, \triangle_3, \triangle_4\}$ is stable. Let ω_{P_9} be the map from $\Gamma(P_9)$ to $\mathbb{Z}$ as in (6) and $\tilde{\omega}_{P_9}$ the map from $\Gamma(P_{10})$ to $\mathbb{Z}$ defined by (2.5) with respect to the $\triangle Y$-exchange at $\triangle_4$. In the following we show $\tilde{\omega}_{P_9} = \omega_{P_{10}}$. Let γ be an element in $\Gamma(P_{10})$. Note that $\Gamma_k(P_{10})$ is the empty set if $k \neq 5, 6, 8, 9$. If γ belongs to $\Gamma_9(P_{10})$, then γ does not contain exactly one of the vertices of P_{10}. Then it is sufficient to consider the following two cases up to symmetry of P_{10}: (i) If γ does not contain

B, then there uniquely exists an element γ' in $\Gamma_9(P_9)$ such that $\bar{\Phi}^{-1}_{P_9,P_{10}}(\gamma) = \{\gamma'\}$. Thus we have $\tilde{\omega}_{P_9}(\gamma) = \omega_{P_9}(\gamma') = 1$. (ii) If γ contains B, then we may assume that γ does not contain A. Then there exists an element γ'_1 in $\Gamma_8(P_9)$ and an element γ'_2 in $\Gamma_9(P_9)$ such that $\bar{\Phi}^{-1}_{P_9,P_{10}}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Since γ'_1 does not contain A, we have $\tilde{\omega}_{P_9}(\gamma) = \omega_{P_9}(\gamma'_1) + \omega_{P_9}(\gamma'_2) = 0 + 1 = 1$. If γ belongs to $\Gamma_8(P_{10})$, then γ does not contain exactly two of the vertices of P_{10}. Then we may assume that γ does not contain either x or w, or γ does not contain either A or w. In the former case, there uniquely exists an element γ' in $\Gamma_7(P_9)$ such that $\bar{\Phi}^{-1}_{P_9,P_{10}}(\gamma) = \{\gamma'\}$. Since γ' contains all of A, v and v', we have $\tilde{\omega}_{P_9}(\gamma) = \omega_{P_9}(\gamma') = 0$. In the latter case, there exists an element γ'_1 in $\Gamma_7(P_9)$ and an element γ'_2 in $\Gamma_8(P_9)$ such that $\bar{\Phi}^{-1}_{P_9,P_{10}}(\gamma) = \{\gamma'_1, \gamma'_2\}$. Since γ'_1 does not contain A and γ'_2 contains all of A, v and v', we have $\tilde{\omega}_{P_9}(\gamma) = \omega_{P_9}(\gamma'_1) + \omega_{P_9}(\gamma'_2) = -1 + 1 = 0$. If γ belongs to $\Gamma_6(P_{10})$, then γ does not contain exactly four of the vertices of P_{10}. Then we may assume that γ does not contain any of A, B, v or v' or γ does not contain any of A, B, z or w. In any case, there uniquely exists an element γ' in $\Gamma_6(P_9)$ such that $\bar{\Phi}^{-1}_{P_9,P_{10}}(\gamma) = \{\gamma'\}$. Since γ' does not contain one of A, v and v', we have $\tilde{\omega}_{P_9}(\gamma) = \omega_{P_9}(\gamma') = -2$. If γ belongs to $\Gamma_5(P_{10})$, then we have $\tilde{\omega}_{P_8}(\gamma) = -1$ in the same way as the case of Q_7. In conclusion, we see that

$$\tilde{\omega}_{P_{10}}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_9(P_{10}) \\ -1 & \text{if } \gamma \in \Gamma_5(P_{10}) \\ -2 & \text{if } \gamma \in \Gamma_6(P_{10}) \\ 0 & \text{otherwise} \end{cases}$$

for an element γ in $\Gamma(P_{10})$. Thus it follows that $\tilde{\omega}_{P_9} = \omega_{P_{10}}$. This completes the proof. □

Remark 3.1. Let us denote the 3-cycle $[Axy]$ of P_7 by $\triangle_6$. Note that P_8 is obtained from P_7 by a single $\triangle Y$-exchange at $\triangle_6$. Let ω_{P_7} be the map from $\Gamma(P_7)$ to $\mathbb{Z}$ as in Theorem 1.2 (4) and $\tilde{\omega}_{P_7}$ the map from $\Gamma(P_8)$ to $\mathbb{Z}$ defined by (2.5) with respect to the $\triangle Y$-exchange at $\triangle_6$. Then it can be shown that $\tilde{\omega}_{P_7}$ coincides with ω_{P_8}.

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